MAHLER MEASURES AS LINEAR COMBINATIONS OF L-VALUES OF MULTIPLE MODULAR FORMS

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Abstract. We study the Mahler measures of certain families of Laurent polynomials in two and three variables. Each of the known Mahler measure formulas for these families involves L-values of at most one newform and/or at most one quadratic character. In this paper, we show, either rigorously or numerically, that the Mahler measures of some polynomials are related to L-values of multiple newforms and quadratic characters simultaneously. The results suggest that the number of modular L-values appearing in the formulas significantly depends on the shape of the algebraic value of the parameter chosen for each polynomial. As a consequence, we also obtain new formulas relating special values of hypergeometric series evaluated at algebraic numbers to special values of L-functions.

1. Introduction

For any Laurent polynomial $P \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$, the Mahler measure of P is defined by

$$m(P) = \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n})| \, d\theta_1 \cdots d\theta_n.$$

(In some parts of the literature, $m(P)$ is called *the logarithmic Mahler measure of* P, but throughout this paper we shall omit the term *logarithmic*.) In general, it is not easy to evaluate Mahler measures of any randomly chosen polynomials if they have more than one variable. Therefore, it is usually difficult to derive explicit formulas of Mahler measures, and it is still unclear what are the precise ways that Mahler measures are related to the polynomials. On the other hand, in some special cases Mahler measures turn out to be related to special values of L-functions attached to the projective varieties arising from the zero loci of the corresponding polynomials. It was Deninger [6] who first discovered this phenomenon using the Bloch-Beilinson conjectures. In particular, he conjectured that the following formula holds:

$$m(x + x^{-1} + y + y^{-1} + 1) = L'(E, 0),$$

where E is the elliptic curve of conductor 15 defined by the projective closure of the zero locus of $x + x^{-1} + y + y^{-1} + 1$. This formula had been conjectural for years before being proved by Rogers and Zudilin [17].

To consider more general situations, we let

$$P_k = x + x^{-1} + y + y^{-1} + k,$$

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where $k \in \mathbb{C}$. It was verified numerically by Boyd [4] that, for many integral values of $k \neq 0, 4$, if E_k is the elliptic curve over $\mathbb{Q}$ determined by the zero locus of P_k, then

$$m(P_k) \stackrel{?}{=} c_k L'(E_k, 0), \tag{1.1}$$

where c_k is a rational number of small height. (Here and throughout $\stackrel{?}{=}$ means that they are equal to at least 25 decimal places.) Note that by the modularity theorem the relation (1.1) is equivalent to

$$m(P_k) \stackrel{?}{=} c_k L'(h_k, 0),$$

where h_k is the newform of weight 2 associated to E_k. (In most situations, we will be dealing with L-values of newforms rather than those of algebraic varieties.) Although Boyd's results seem to be highly accurate, rigorous proofs of these formulas are quite rare. Inspired by these results, Rodriguez Villegas [12] proved that $m(P_k)$ can be expressed in terms of Eisenstein-Kronecker series, and for certain values of k they turn out to be related to special values of L-series of elliptic curves with complex multiplication. He also observed from his numerical data that the relation (1.1) seems to hold for every sufficiently large k such that $k^2 \in \mathbb{N}$. For instance, he proved that

$$m(P_{4\sqrt{2}}) = L'(E_{4\sqrt{2}}, 0) = L'(f_{64}, 0), \tag{1.2}$$

$$m\left(P_{\frac{4}{\sqrt{2}}}\right) = L'\left(E_{\frac{4}{\sqrt{2}}}, 0\right) = L'(f_{32}, 0), \tag{1.3}$$

where f_{64} and f_{32} are newforms of weight 2 and level 64 and 32, associated to the elliptic curves $E_{4\sqrt{2}}$ and $E_{\frac{4}{\sqrt{2}}}$, respectively. Similar results, due to Lalín and Rogers, can be found in [10]. In Section 2 we will deduce formulas for $m(P_k)$ when $k = \sqrt{8 \pm 6\sqrt{2}}$. Indeed, we will prove that

$$m\left(P_{\sqrt{8\pm 6\sqrt{2}}}\right) = \frac{1}{2}\left(L'(f_{64}, 0) \pm L'(f_{32}, 0)\right). \tag{1.4}$$

Using similar arguments one obtains conjectured formulas in terms of two different L-values for $m(P_k)$ when $k = \sqrt{8 \pm 9\sqrt{2}}$. Observe that in these cases $k^2 \notin \mathbb{Z}$, so it is not surprising that our results are somewhat different from those of Rodriguez Villegas. In addition, we consider the Hesse family

$$Q_k = x^3 + y^3 + 1 - kxy.$$

This family was also investigated in [12], and it was pointed out that the Mahler measures of Q_k appear to be of the form (1.1) when k is sufficiently large and $k^3 \in \mathbb{Z}$. On the other hand, we will prove that if $k = \sqrt[3]{6 - 6\sqrt[3]{2} + 18\sqrt[3]{4}}$, then

$$m(Q_k) = \frac{1}{2}\left(L'(f_{108}, 0) + L'(f_{36}, 0) - 3L'(f_{27}, 0)\right),$$

where f_N is a newform of weight 2 and level N.

In Section 3 we will establish some formulas concerning three-variable Mahler measures. The author showed in [18] that for many values of k the Mahler measures of the following Laurent polynomials:

$$(x + x^{-1})(y + y^{-1})(z + z^{-1}) + k,$$
$$(x + x^{-1})^2(y + y^{-1})^2(1 + z)^3 z^{-2} - k,$$
$$x^4 + y^4 + z^4 + 1 + kxyz$$

are of the form

$$m(P) = c_1 L'(g,0) + c_2 L'(\chi,-1) \tag{1.5}$$

for some weight 3 newform g, an odd quadratic character χ, and $c_1, c_2 \in \mathbb{Q}$. To obtain the formulas of type (1.5), it seems that the chosen value of k necessarily satisfies similar conditions as observed in the two-variable case. For instance, for the last family, k must be sufficiently large and $k^4 \in \mathbb{Z}$. We will give some examples of Mahler measures of polynomials in this family when k^4 are algebraic integers but not rational integers, which reveal similar phenomena as seen in the two-variable case. For example, it will be proved that when $k = \sqrt[4]{26856 + 15300\sqrt{3}}$ the following equality is true:

$$m(x^4+y^4+z^4+1+kxyz) = \frac{5}{48}\left(20L'(g_{12},0) + 4L'(g_{48},0) + 11L'(\chi_{-3},-1) + 8L'(\chi_{-4},-1)\right),$$

where g_N is a newform of weight 3 and level N and $\chi_D(n) = \left(\frac{D}{n}\right)$.

In Section 4 we establish a functional equation of some other three-variable Mahler measures, which gives us a five-term relation between the Mahler measures with algebraic arguments. We also give an explicit example which is related to multiple special L-values. Many parts of this problem are still wide open and can be done further in several directions.

One of the things that all families mentioned above have in common is that their Mahler measures can be written in terms of hypergeometric series. Therefore, one can deduce some interesting hypergeometric evaluations from Mahler measure formulas easily. For instance, the equality (1.4) implies that

$${}_4F_3\left(\begin{matrix}\frac{3}{2}, \frac{3}{2}, 1, 1 \\ 2, 2, 2\end{matrix}; -16+12\sqrt{2}\right) = \frac{4+3\sqrt{2}}{2}\left(\log(8+6\sqrt{2}) - (L'(f_{64},0) + L'(f_{32},0))\right).$$

By some numerical computations in `Maple` and `Sage`, we also discover a number of conjectured formulas for the three-variable Mahler measures studied in [14] and [18] which involve several L-values, as listed at the end of this paper.

2. Two-variable Mahler measures

As mentioned earlier, we will study Mahler measures of the two families with the complex parameter t, namely

$$\begin{aligned} m_2(t) &:= 2m(P_{t^{1/2}}) = 2m(x+x^{-1}+y+y^{-1}+t^{1/2}), \\ m_3(t) &:= 3m(Q_{t^{1/3}}) = 3m(x^3+y^3+1-t^{1/3}xy). \end{aligned}$$

It is known that for most values of t the Mahler measures $m_2(t)$ and $m_3(t)$ can be expressed in terms of hypergeometric series. Indeed, we have the following result: (See, for instance, [13, Thm. 3.1].)

Theorem 2.1. *Let $m_2(t)$ and $m_3(t)$ be as defined above.*

(i) *If $t \neq 0$, then* $m_2(t) = \operatorname{Re}\left(\log(t) - \frac{4}{t}\,{}_4F_3\left(\begin{matrix}\frac{3}{2}, \frac{3}{2}, 1, 1 \\ 2, 2, 2\end{matrix}; \frac{16}{t}\right)\right).$

(ii) *If $|t|$ is sufficiently large, then* $m_3(t) = \operatorname{Re}\left(\log(t) - \frac{6}{t}\,{}_4F_3\left(\begin{matrix}\frac{4}{3}, \frac{5}{3}, 1, 1 \\ 2, 2, 2\end{matrix}; \frac{27}{t}\right)\right).$

Furthermore, Kurokawa and Ochiai [9] and Lalín and Rogers [10] showed that $m_2(t)$ satisfies some functional equations, which enable us to prove and to conjecture new Mahler measure formulas for some $t \notin \mathbb{Z}$. Throughout this section, f_N denotes a normalized newform of weight 2 and level N with rational Fourier coefficients.

Proposition 2.2. *The following identities are true:*

$$m_2(8+6\sqrt{2}) = L'(f_{64},0) + L'(f_{32},0), \tag{2.3}$$

$$m_2(8-6\sqrt{2}) = L'(f_{64},0) - L'(f_{32},0), \tag{2.4}$$

where $f_{64}(\tau) = \dfrac{\eta^8(8\tau)}{\eta^2(4\tau)\eta^2(16\tau)} \in S_2(\Gamma_0(64))$ *and* $f_{32}(\tau) = \eta^2(4\tau)\eta^2(8\tau) \in S_2(\Gamma_0(32))$. *(As usual, let* η *denote the Dedekind eta function,*

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n),$$

where $q = e^{2\pi i \tau}$, *and let* $S_k(\Gamma_0(N))$ *denote the space of cusp forms of weight* k *and level* N.*)*

Proof. It was proved in [9, Thm. 7] that if $k \in \mathbb{R}\backslash\{0\}$, then

$$2m_2\left(4\left(k+\frac{1}{k}\right)^2\right) = m_2(16k^4) + m_2\left(\frac{16}{k^4}\right). \tag{2.5}$$

Recall from (1.2) and (1.3) that $m_2(32) = 2L'(f_{64},0)$ and $m_2(8) = 2L'(f_{32},0)$, so we can deduce (2.3) easily by substituting $k = 2^{1/4}$ in (2.5). On the other hand, one sees from [10, Thm. 2.2] that the following functional equation holds for any k such that $0 < |k| < 1$:

$$m_2\left(4\left(k+\frac{1}{k}\right)^2\right) + m_2\left(-4\left(k-\frac{1}{k}\right)^2\right) = m_2\left(\frac{16}{k^4}\right). \tag{2.6}$$

In particular, choosing $k = 2^{-1/4}$, we obtain

$$m_2(8+6\sqrt{2}) + m_2(8-6\sqrt{2}) = m_2(32).$$

Now (2.4) follows immediately from the known information above. $\square$

Rodriguez Villegas [12, Table 4] verified numerically that $m_2(128) \stackrel{?}{=} \frac{1}{2}L'(f_{448},0)$ and $m_2(2) \stackrel{?}{=} \frac{1}{2}L'(f_{56},0)$, where $f_{448}(\tau) = q - 2q^5 - q^7 - 3q^9 + 4q^{11} - 2q^{13} - 6q^{17} - \cdots$ and $f_{56}(\tau) = q + 2q^5 - q^7 - 3q^9 - 4q^{11} + 2q^{13} - 6q^{17} + \cdots$. Therefore, letting $k = 2^{3/4}$ in (2.5) and $k = 2^{-3/4}$ in (2.6) results in a couple of conjectured formulas similar to (2.3) and (2.4).

Conjecture 2.7. *The following identities are true:*

$$m_2\left(8+9\sqrt{2}\right) \stackrel{?}{=} \frac{1}{4}\left(L'(f_{448},0) + L'(f_{56},0)\right),$$

$$m_2\left(8-9\sqrt{2}\right) \stackrel{?}{=} \frac{1}{4}\left(L'(f_{448},0) - L'(f_{56},0)\right).$$

We also found via numerical computations the following conjectured formulas:

$$m_2\left(\frac{49+9\sqrt{17}}{2}\right) \stackrel{?}{=} \frac{1}{2}\left(L'(f_{289},0)+8L'(f_{17},0)\right),$$

$$m_2\left(\frac{49-9\sqrt{17}}{2}\right) \stackrel{?}{=} \frac{1}{2}\left(L'(f_{289},0)-8L'(f_{17},0)\right),$$

where $f_{289}(\tau)=q-q^2-q^4+2q^5-4q^7+3q^8-3q^9-\cdots$ and $f_{17}(\tau)=q-q^2-q^4-2q^5+4q^7+3q^8-3q^9+\cdots$. Observe that we can again employ the identity (2.5) for $k=(1+\sqrt{17})/4$ to deduce

$$2m_2(17)=m_2\left(\frac{49+9\sqrt{17}}{2}\right)+m_2\left(\frac{49-9\sqrt{17}}{2}\right) \stackrel{?}{=} L'(f_{289},0),$$

which is equivalent to a conjectured formula in [12, Table 4]. A weaker form of these formulas, namely

$$m_2\left(\frac{49+9\sqrt{17}}{2}\right)-m_2(17) \stackrel{?}{=} 4L'(f_{17},0),$$

was also briefly discussed in [15, §4].

To study the Mahler measure $m_3(t)$, we use the following crucial result, which basically states that $m_3(t)$ can be written in terms of Eisenstein-Kronecker series when t is parameterized properly.

Theorem 2.8 (Rodriguez Villegas [12, §IV])**.** *Let $t_3(\tau)=27+\left(\frac{\eta(\tau)}{\eta(3\tau)}\right)^{12}$, and let $\mathcal{F}$ be the fundamental domain for $\Gamma_0(3)$ with vertices $i\infty, 0, (1+i/\sqrt{3})/2$, and $(-1+i/\sqrt{3})/2$. If $\tau\in\mathcal{F}$, then*

$$m_3(t_3(\tau))=\frac{81\sqrt{3}\,\mathrm{Im}(\tau)}{4\pi^2}\sideset{}{'}\sum_{m,n\in\mathbb{Z}}\frac{\chi_{-3}(m)(m+3n\,\mathrm{Re}(\tau))}{[(m+3n\tau)(m+3n\bar{\tau})]^2},$$

where $\sideset{}{'}\sum_{m,n}$ means that $(m,n)=(0,0)$ is excluded from the summation.

The remaining part of this section will be devoted to proving the following result:

Theorem 2.9. *If $t=6-6\sqrt[3]{2}+18\sqrt[3]{4}$, then*

$$m_3(t)=\frac{3}{2}\left(L'(f_{108},0)+L'(f_{36},0)-3L'(f_{27},0)\right),$$

where $f_{36}(\tau)=\eta^4(6\tau)\in S_2(\Gamma_0(36))$, $f_{27}(\tau)=\eta^2(3\tau)\eta^2(9\tau)\in S_2(\Gamma_0(27))$, and $f_{108}(\tau)=q+5q^7-7q^{13}-q^{19}-5q^{25}-4q^{31}-q^{37}+\cdots$, the unique normalized newform in $S_2(\Gamma_0(108))$.

Applying Theorem 2.1, Proposition 2.2, and Theorem 2.9, one obtains the following hypergeometric evaluation formulas immediately:

Corollary 2.10. *The following identities hold:*

$$
{}_4F_3\left(\begin{matrix}\frac{3}{2},\ \frac{3}{2},\ 1,\ 1\\ 2,\ 2,\ 2\end{matrix};-16+12\sqrt{2}\right)=\frac{4+3\sqrt{2}}{2}\left(\log(8+6\sqrt{2})-(L'(f_{64},0)+L'(f_{32},0))\right),
$$

$$
\begin{aligned}
{}_4F_3\left(\begin{matrix}\frac{4}{3},\ \frac{5}{3},\ 1,\ 1\\ 2,\ 2,\ 2\end{matrix};\frac{63+171\sqrt[3]{2}-18\sqrt[3]{4}}{250}\right)&=\left(1-\sqrt[3]{2}+3\sqrt[3]{4}\right)\Big(\log(6-6\sqrt[3]{2}+18\sqrt[3]{4})\\
&\quad-\frac{3}{2}\left(L'(f_{108},0)+L'(f_{36},0)-3L'(f_{27},0)\right)\Big).
\end{aligned}
$$

To establish Theorem 2.9, we require some identities for L-values of the involved cusp forms, which will be verified in the following lemmas.

Lemma 2.11. *Let $f_{36}(\tau)$ be as defined in Theorem 2.9. Then the following equality holds:*

$$
L(f_{36},2)=\frac{1}{2}\sum_{m,n\in\mathbb{Z}}{}^{'}\frac{m\chi_{-3}(m)}{(m^2+3n^2)^2}.
$$

Proof. First, note that for any τ in the upper half plane $\eta(\tau)$ satisfies the functional equation

$$
\eta\left(-\frac{1}{\tau}\right)=\sqrt{-i\tau}\,\eta(\tau).
$$

Hence it is easily seen that

$$
\frac{\eta\left(\frac{\sqrt{-3}}{3}\right)}{\eta\left(\sqrt{-3}\right)}=3^{\frac{1}{4}},
$$

which implies that $t_3\left(\frac{\sqrt{-3}}{3}\right)=54$. Thus we have from Theorem 2.8 that

$$
m_3(54)=\frac{81}{4\pi^2}\sum_{m,n\in\mathbb{Z}}{}^{'}\frac{m\chi_{-3}(m)}{(m^2+3n^2)^2}.
$$

On the other hand, Rogers [13, Thm. 2.1, Thm. 5.2] proved that

$$
m_3(54)=\frac{81}{2\pi^2}L(f_{36},2),
$$

whence the lemma follows. □

Lemma 2.12. *Let $f_{108}(\tau)$ be the unique normalized newform with rational coefficients in $S_2(\Gamma_0(108))$, and let $\mathcal{A}=\{(m,n)\in\mathbb{Z}^2\mid(m,n)\equiv(-1,-2),(2,1),(1,0),(-2,3)\mod 6\}$. Then*

$$
L(f_{108},2)=\sum_{m,n\in\mathcal{A}}\frac{m+3n}{(m^2+3n^2)^2}.
$$

Proof. By taking the Mellin transform of the newform, it suffices to prove that

$$
f_{108}(\tau)=\sum_{m,n\in\mathcal{A}}(m+3n)q^{m^2+3n^2}. \tag{2.13}
$$

Let $K=\mathbb{Q}(\sqrt{-3}),\mathcal{O}_K=\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right],\Lambda=(3+3\sqrt{-3})\subset\mathcal{O}_K$, and $I(\Lambda)=$ the group of fractional ideals of $\mathcal{O}_K$ coprime to Λ. Since Λ can be factorized as

$$\Lambda=\left(\frac{1+\sqrt{-3}}{2}\right)(\sqrt{-3})^2(2),$$

any integral ideal $\mathfrak{a}$ is coprime to Λ if and only if $(\sqrt{-3})\nmid\mathfrak{a}$ and $(2)\nmid\mathfrak{a}$. As a consequence, every integral ideal coprime to Λ is uniquely represented by $(m+n\sqrt{-3})$, where $m,n\in\mathbb{Z},m>0,3\nmid m$, and $m\not\equiv n\pmod 2$. Let $P(\Lambda)$ denote the monoid of integral ideals coprime to Λ.

Define $\varphi:P(\Lambda)\to\mathbb{C}^\times$ by

$$\varphi((m+n\sqrt{-3}))=\begin{cases}\dfrac{-\chi_{-3}(m)m+\chi_{-3}(n)(3n)-(\chi_{-3}(n)m+\chi_{-3}(m)n)\sqrt{-3}}{2} & \text{if } 3\nmid n,\\ \chi_{-3}(m)(m+n\sqrt{-3}) & \text{if } 3|n.\end{cases}$$

Then it is not difficult to check that φ is multiplicative, and for each $(m+n\sqrt{-3})\in P(\Lambda)$ with $m+n\sqrt{-3}\equiv 1\pmod{\Lambda}$,

$$\varphi((m+n\sqrt{-3}))=m+n\sqrt{-3}.$$

Hence we can extend φ multiplicatively to define a Hecke Grössencharacter of weight 2 and conductor Λ on $I(\Lambda)$. Now if we let

$$\Psi(\tau):=\sum_{\mathfrak{a}\in P(\Lambda)}\varphi(\mathfrak{a})q^{N(\mathfrak{a})},$$

then one sees from [11, Thm. 1.31] that $\Psi(\tau)$ is a newform in $S_2(\Gamma_0(108))$. Observe that

$$\varphi((m+n\sqrt{-3}))+\varphi((m-n\sqrt{-3}))=\begin{cases}-\chi_{-3}(m)m+\chi_{-3}(n)(3n) & \text{if } 3\nmid n,\\ 2\chi_{-3}(m)m & \text{if } 3|n,\end{cases}$$

so we have

$$\Psi(\tau)=\sum_{\substack{m,n\in\mathbb{N}\\ 3\nmid m,3\nmid n\\ m\not\equiv n\pmod 2}}(-\chi_{-3}(m)m+\chi_{-3}(n)(3n))q^{m^2+3n^2}+\sum_{\substack{m\in\mathbb{N},n\in\mathbb{Z}\\ 3\nmid m,3|n\\ m\not\equiv n\pmod 2}}\chi_{-3}(m)mq^{m^2+3n^2}.$$

Working modulo 6, one can show that

$$\sum_{\substack{m,n\in\mathbb{N}\\ 3\nmid m,3\nmid n\\ m\not\equiv n\pmod 2}}(-\chi_{-3}(m)m+\chi_{-3}(n)(3n))q^{m^2+3n^2}=\sum_{\substack{m,n\in\mathbb{Z}\\ (m,n)\equiv(-1,2),(2,1)\\ \pmod 6}}(m+3n)q^{m^2+3n^2},$$

and

$$\begin{aligned}\sum_{\substack{m\in\mathbb{N},n\in\mathbb{Z}\\ 3\nmid m,3|n\\ m\not\equiv n\pmod 2}}\chi_{-3}(m)mq^{m^2+3n^2}&=\sum_{\substack{m,n\in\mathbb{Z}\\ (m,n)\equiv(1,0),(-2,3)\\ \pmod 6}}mq^{m^2+3n^2}\\ &=\sum_{\substack{m,n\in\mathbb{Z}\\ (m,n)\equiv(1,0),(-2,3)\\ \pmod 6}}(m+3n)q^{m^2+3n^2}.\end{aligned}$$

Consequently, the coefficients of $\Psi(\tau)$ are rational, which implies that $\Psi(\tau) = f_{108}(\tau)$, and (2.13) holds. (One can check using, for example, `Sage` or `Magma` that there is only one normalized newform in $S_2(\Gamma_0(108))$.) $\square$

Lemma 2.14. *Let $f_{27}(\tau)$ be as defined in Theorem 2.9, and let $\mathcal{B} = \{(m,n) \in \mathbb{Z}^2 \mid (m,n) \equiv (1,0),(-2,3),(1,-1),(-2,2),(2,-1),(-1,2) \mod 6\}$. Then*

$$L(f_{27},2) = {\sum_{m,n\in\mathcal{B}}}' \frac{m+3n}{(m^2+3n^2)^2}.$$

Proof. As before, we will establish a q-expansion for $f_{27}(\tau)$ first; i.e., we aim at proving that

$$f_{27}(\tau) = \sum_{m,n\in\mathcal{B}} (m+3n)q^{m^2+3n^2}.$$

Recall from [13, §6] that the following identity is true:

$$f_{27}(\tau) = \sum_{\substack{m,n\in\mathbb{Z}\\ (m,n)\equiv(1,1),(-2,-2)\\ (\text{mod } 6)}} \left(\frac{m+3n}{4}\right) q^{\frac{m^2+3n^2}{4}}. \tag{2.15}$$

Therefore, it is sufficient to prove the following claims, each of which involves only simple manipulation. (Unless otherwise stated, each ordered pair (a,b) listed beneath the sigma sign indicates all $(m,n) \in \mathbb{Z}^2$ such that $m \equiv a$ and $n \equiv b \pmod 6$.)

Claim 1.

$$\sum_{(1,1)} \left(\frac{m+3n}{4}\right) q^{\frac{m^2+3n^2}{4}} = \sum_{(1,0),(-2,3)} (m+3n)q^{m^2+3n^2} + \sum_{(2,-1),(-1,2)} \left(\frac{m+3n}{2}\right) q^{m^2+3n^2}.$$

Claim 2.

$$\sum_{(-2,-2)} \left(\frac{m+3n}{4}\right) q^{\frac{m^2+3n^2}{4}} = \sum_{(1,-1),(-2,2)} (m+3n)q^{m^2+3n^2} + \sum_{(2,-1),(-1,2)} \left(\frac{m+3n}{2}\right) q^{m^2+3n^2}.$$

Proof of Claim 1. It is clear that

$$\begin{aligned}\sum_{(1,0),(-2,3)} (m+3n)q^{m^2+3n^2} &= \sum_{(1,0),(-2,3)} mq^{m^2+3n^2}\\ &= \sum_{(1,0),(-2,3)} \left(\frac{(m+3n)+3(m-n)}{4}\right) q^{\frac{(m+3n)^2+3(m-n)^2}{4}}, \text{ and}\end{aligned}$$

$$\sum_{(2,-1),(-1,2)} \left(\frac{m+3n}{2}\right) q^{m^2+3n^2} = \sum_{(2,-1),(-1,2)} \left(\frac{(3n-m)+3(m+n)}{4}\right) q^{\frac{(3n-m)^2+3(m+n)^2}{4}}.$$

Also, it can be verified in a straightforward manner that

$$\begin{aligned}\{(m,n) \mid m \equiv n \equiv 1 \pmod 6\} = &\{(k+3l, k-l) \mid (k,l) \equiv (1,0),(-2,3) \pmod 6\}\\ &\uplus\{(3l-k, k+l) \mid (k,l) \equiv (2,-1),(-1,2) \pmod 6\},\end{aligned}$$

where $\uplus$ denotes disjoint union, so we obtain Claim 1.

Proof of Claim 2. Let us make some observation first that, by symmetry,

$$\sum_{(1,-1),(-2,2)} (3m+3n)q^{m^2+3n^2} = 0,$$

so we have that

$$\sum_{(1,-1),(-2,2)} (-2m)q^{m^2+3n^2} = \sum_{(1,-1),(-2,2)} (m+3n)q^{m^2+3n^2}.$$

It follows that

$$\begin{aligned}
\sum_{(-1,-1),(2,2)} (m+3n)q^{m^2+3n^2} &= \sum_{(1,-1),(-2,2)} (-m+3n)q^{m^2+3n^2} \\
&= \sum_{(1,-1),(-2,2)} (m+3n)q^{m^2+3n^2} + \sum_{(1,-1),(-2,2)} (-2m)q^{m^2+3n^2} \qquad (2.16) \\
&= 2\sum_{(1,-1),(-2,2)} (m+3n)q^{m^2+3n^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{(-2,-2)} \left(\frac{m+3n}{4}\right) q^{\frac{m^2+3n^2}{4}} &= \sum_{(-1,-1) \pmod 3} \left(\frac{m+3n}{2}\right) q^{m^2+3n^2} \\
&= \sum_{\substack{(-1,-1),(2,2)\\(2,-1),(-1,2)}} \left(\frac{m+3n}{2}\right) q^{m^2+3n^2} \\
&= \sum_{(1,-1),(-2,2)} (m+3n)q^{m^2+3n^2} + \sum_{(2,-1),(-1,2)} \left(\frac{m+3n}{2}\right) q^{m^2+3n^2},
\end{aligned}$$

where the last equality comes from (2.16).

□

Lemma 2.17. *The following equality is true:*

$$L(f_{108},2) - \frac{3}{4}L(f_{27},2) = \frac{3}{2}\sum_{\substack{m,n\in\mathbb{Z}\\3\nmid n}} \frac{m\chi_{-3}(m)}{(3m^2+n^2)^2}.$$

Proof. Taking the Mellin transform of $f_{27}(\tau)$ in (2.15) yields

$$L(f_{27},2) = 4\sum_{(1,1),(-2,-2)} \frac{m+3n}{(3m^2+n^2)^2}. \qquad (2.18)$$

Since $\chi_{-3}(n) = j$ iff $n \equiv j \pmod 3$, where $j \in \{-1,0,1\}$, we have that

$$\begin{aligned}
\sum_{\substack{m,n\in\mathbb{Z}\\3\nmid n}} \frac{m\chi_{-3}(m)}{(3m^2+n^2)^2} &= \sum_{\substack{m,n\in\mathbb{Z}\\3\nmid m}} \frac{n\chi_{-3}(n)}{(m^2+3n^2)^2} \\
&= \sum_{\substack{n\equiv -1 \pmod 3\\3\nmid m}} \frac{-2n}{(m^2+3n^2)^2}.
\end{aligned}$$

Also, it is obvious that the symmetry of the summation yields

$$\sum_{\substack{n\equiv -1 \pmod 3\\ 3\nmid m}} \frac{m}{(m^2+3n^2)^2} = 0.$$

Hence, using Lemma 2.12, one sees that

$$\begin{aligned}
L(f_{108},2) - \frac{3}{2}\sum_{\substack{m,n\in\mathbb{Z}\\ 3\nmid n}} \frac{m\chi_{-3}(m)}{(3m^2+n^2)^2} &= \sum_{\substack{(-1,-2),(2,1)\\ (1,0),(-2,3)}} \frac{m+3n}{(m^2+3n^2)^2} + \sum_{\substack{n\equiv -1 \pmod 3\\ 3\nmid m}} \frac{3n}{(m^2+3n^2)^2}\\
&= \sum_{\substack{(-1,-2),(2,1)\\ (1,0),(-2,3)}} \frac{m+3n}{(m^2+3n^2)^2} + \sum_{\substack{n\equiv -1 \pmod 3\\ 3\nmid m}} \frac{m+3n}{(m^2+3n^2)^2}\\
&= \sum_{\substack{(-1,-2),(2,1)\\ (1,0),(-2,3)}} \frac{m+3n}{(m^2+3n^2)^2} + \sum_{\substack{(-2,2),(-2,-1)\\ (-1,2),(-1,-1)\\ (1,2),(1,-1)\\ (2,2),(2,-1)}} \frac{m+3n}{(m^2+3n^2)^2}\\
&= \sum_{\substack{(1,0),(-2,3)\\ (1,-1),(-2,2)\\ (2,-1),(-1,2)}} \frac{m+3n}{(m^2+3n^2)^2} - \sum_{(1,1),(-2,-2)} \frac{m+3n}{(m^2+3n^2)^2}\\
&= L(f_{27},2) - \frac{1}{4}L(f_{27},2),
\end{aligned}$$

where we have applied Lemma 2.14 and (2.18) in the last equality. □

Putting the previous lemmas together, we are now ready to complete a proof of Theorem 2.9.

Proof of Theorem 2.9. Let $\tau_0 = \sqrt{-3}/9$. Then $t_3(\tau_0) = 6 - 6\sqrt[3]{2} + 18\sqrt[3]{4}$. This can be verified by considering numerical approximation of $t_3(\tau_0)$ and using the following identities:

$$\begin{aligned}
j(\tau) &= j(-1/\tau), \qquad \mathfrak{f}^3(\sqrt{-27}) = 2(1+\sqrt[3]{2}+\sqrt[3]{4}),\\
j(\tau) &= \frac{(\mathfrak{f}^{24}(\tau)-16)^3}{\mathfrak{f}^{24}(\tau)} = \frac{t_3(\tau)(t_3(\tau)+216)^3}{(t_3(\tau)-27)^3},
\end{aligned}$$

where $j(\tau)$ is the j-invariant, and $\mathfrak{f}(\tau)$ is a Weber modular function defined by

$$\mathfrak{f}(\tau) = e^{-\frac{\pi i}{24}} \frac{\eta\left(\frac{\tau+1}{2}\right)}{\eta(\tau)}.$$

(For references to these identities, see [5, §1], [19, Table VI], and [20, §1].) Then we see from Theorem 2.8 that

$$\begin{aligned}
m_3(t_3(\tau_0)) &= \frac{27}{4\pi^2}\sum_{m,n\in\mathbb{Z}}{}' \frac{m\chi_{-3}(m)}{(m^2+\frac{n^2}{3})^2} \\
&= \frac{3}{2}\left(\frac{81}{2\pi^2}\sum_{m,n\in\mathbb{Z}}{}' \frac{m\chi_{-3}(m)}{(3m^2+n^2)^2}\right) \\
&= \frac{3}{2}\left(\frac{81}{2\pi^2}\sum_{\substack{m,n\in\mathbb{Z}\\ 3\mid n}}{}' \frac{m\chi_{-3}(m)}{(3m^2+n^2)^2} + \frac{81}{2\pi^2}\sum_{\substack{m,n\in\mathbb{Z}\\ 3\nmid n}} \frac{m\chi_{-3}(m)}{(3m^2+n^2)^2}\right) \\
&= \frac{3}{2}\left(\frac{9}{2\pi^2}\sum_{m,n\in\mathbb{Z}}{}' \frac{m\chi_{-3}(m)}{(m^2+3n^2)^2} + \frac{81}{2\pi^2}\sum_{\substack{m,n\in\mathbb{Z}\\ 3\nmid n}} \frac{m\chi_{-3}(m)}{(3m^2+n^2)^2}\right).
\end{aligned}$$

Now we can deduce using Lemma 2.11 and Lemma 2.17 that

$$m_3(t_3(\tau_0)) = \frac{3}{2}\left(\frac{27}{\pi^2}L(f_{108},2) + \frac{9}{\pi^2}L(f_{36},2) - \frac{81}{4\pi^2}L(f_{27},2)\right). \tag{2.19}$$

Finally, the formula stated in the theorem is merely a simple consequence of (2.19) and the functional equation

$$\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(f,s) = \epsilon\left(\frac{\sqrt{N}}{2\pi}\right)^{2-s}\Gamma(2-s)L(f,2-s),$$

where f is any newform of weight 2 and level N with real Fourier coefficients, and $\epsilon \in \{-1,1\}$, depending on f. (If $f \in \{f_{27}, f_{36}, f_{108}\}$, then $\epsilon = 1$.) □

In addition to the formula stated in Theorem 2.9, we discovered some other conjectured formulas of similar type using numerical values of the hypergeometric representation of $m_3(t)$ given in Theorem 2.1:

$$\begin{aligned}
m_3\left(17766+14094\sqrt[3]{2}+11178\sqrt[3]{4}\right) &\stackrel{?}{=} \frac{3}{2}(L'(f_{108},0)+3L'(f_{36},0)+3L'(f_{27},0)), \\
m_3(\alpha\pm\beta i) &\stackrel{?}{=} \frac{3}{2}(L'(f_{108},0)+3L'(f_{36},0)-6L'(f_{27},0)), \\
m_3\left(\frac{(7+\sqrt{5})^3}{4}\right) &\stackrel{?}{=} \frac{1}{8}\left(9L'(f_{100},0)+38L'(f_{20},0)\right), \\
m_3\left(\frac{(7-\sqrt{5})^3}{4}\right) &\stackrel{?}{=} \frac{1}{4}\left(9L'(f_{100},0)-38L'(f_{20},0)\right),
\end{aligned}$$

where $\alpha = 17766-7047\sqrt[3]{2}-5589\sqrt[3]{4}$, $\beta = 27\sqrt{3}(261\sqrt[3]{2}-207\sqrt[3]{4})$, $f_{100}(\tau) = q+2q^3-2q^7+q^9-2q^{13}+6q^{17}-4q^{19}-\cdots$, and $f_{20}(\tau) = \eta^2(2\tau)\eta^2(10\tau)$.

It is worth mentioning that the last two Mahler measures above also appear in [8, Thm. 6] and [16, §4]. More precisely, it was shown that

$$19m_3(32) = 16m_3\left(\frac{(7+\sqrt{5})^3}{4}\right) - 8m_3\left(\frac{(7-\sqrt{5})^3}{4}\right), \tag{2.20}$$

$$m_3(32) = 8L'(f_{20},0). \tag{2.21}$$

Many of the identities like (2.20) can be proved using the elliptic dilogarithm evaluated at some torsion points on the corresponding elliptic curve. However, to our knowledge, no rigorous proof of the conjectured formulas for the individual terms on the right seems to appear in the literature.

3. Three-variable Mahler measures

From here on, we denote

$$\begin{aligned}
n_2(s) &:= 2m((x+x^{-1})(y+y^{-1})(z+z^{-1})+s^{1/2}),\\
n_3(s) &:= m\left((x+x^{-1})^2(y+y^{-1})^2(1+z)^3z^{-2}-s\right),\\
n_4(s) &:= 4m(x^4+y^4+z^4+1+s^{1/4}xyz),\\
s_2(\tau) &:= -\frac{\Delta\left(\tau+\frac{1}{2}\right)}{\Delta(2\tau+1)},\\
s_3(\tau) &:= \left(27\left(\frac{\eta(3\tau)}{\eta(\tau)}\right)^6+\left(\frac{\eta(\tau)}{\eta(3\tau)}\right)^6\right)^2,\\
s_4(\tau) &:= \frac{\Delta(2\tau)}{\Delta(\tau)}\left(16\left(\frac{\eta(\tau)\eta(4\tau)^2}{\eta(2\tau)^3}\right)^4+\left(\frac{\eta(2\tau)^3}{\eta(\tau)\eta(4\tau)^2}\right)^4\right)^4,
\end{aligned}$$

where $\Delta(\tau)=\eta^{24}(\tau)$.

The main result we will show in this section is stated as follows:

Proposition 3.1. *The following identities are true:*

$$n_4(26856+15300\sqrt{3}) = \frac{5}{12}\left(20L'(g_{12},0)+4L'(g_{48},0)+11L'(\chi_{-3},-1)+8L'(\chi_{-4},-1)\right),$$

$$n_4(26856-15300\sqrt{3}) = \frac{5}{6}\left(-20L'(g_{12},0)+4L'(g_{48},0)-11L'(\chi_{-3},-1)+8L'(\chi_{-4},-1)\right),$$

where $g_{12}(\tau)=\eta^3(2\tau)\eta^3(6\tau)\in S_3(\Gamma_0(12),\chi_{-3})$ *and* $g_{48}(\tau)=\dfrac{\eta^9(4\tau)\eta^9(12\tau)}{\eta^3(2\tau)\eta^3(6\tau)\eta^3(8\tau)\eta^3(24\tau)}\in S_3(\Gamma_0(48),\chi_{-3})$.

Proof. By a result in [18, Prop. 2.1], we have that $n_4(s)$ can be expressed as Eisenstien-Kronecker series when s is parameterized by $s_4(\tau)$, namely

$$\begin{aligned}
n_4(s_4(\tau)) = \frac{10\operatorname{Im}(\tau)}{\pi^3}\sideset{}{'}\sum_{m,n\in\mathbb{Z}}\Bigg(&-\left(\frac{4n^2}{(m^2|\tau|^2+n^2)^3}-\frac{1}{(m^2|\tau|^2+n^2)^2}\right)\\
&+4\left(\frac{4n^2}{(4m^2|\tau|^2+n^2)^3}-\frac{1}{(4m^2|\tau|^2+n^2)^2}\right)\Bigg)
\end{aligned} \tag{3.2}$$

for every $\tau \in \mathbb{C}$ such that τ is purely imaginary and $\operatorname{Im}(\tau) \geq 1/\sqrt{2}$. It is clear that $s_4(\tau)$ can be rewritten in the form

$$s_4(\tau) = \frac{1}{\mathfrak{f}_1^8(2\tau)}\left(\frac{16}{\mathfrak{f}_1^8(4\tau)} + \frac{\mathfrak{f}_1^8(4\tau)}{\mathfrak{f}_1^8(2\tau)}\right)^4,$$

where $\mathfrak{f}_1(\tau) := \dfrac{\eta\left(\frac{\tau}{2}\right)}{\eta(\tau)}$, also known as a Weber modular function. We obtain from [19, Table VI] that

$$\mathfrak{f}_1^4(\sqrt{-12}) = 2^{\frac{7}{6}}(1+\sqrt{3}), \qquad \mathfrak{f}_1^8(\sqrt{-48}) = 2^{\frac{19}{6}}(1+\sqrt{3})(\sqrt{2}+\sqrt{3})^2(1+\sqrt{2})^2.$$

Therefore, after simplifying, we have $s_4\left(\sqrt{-3}\right) = 26856 + 15300\sqrt{3}$, and substituiting $\tau = \sqrt{-3}$ in (3.2) yields

$$\begin{aligned} n_4(26856+15300\sqrt{3}) &= \frac{10\sqrt{3}}{\pi^3}\sum_{m,n\in\mathbb{Z}}{}' \Bigg(-\left(\frac{4n^2}{(3m^2+n^2)^3} - \frac{1}{(3m^2+n^2)^2}\right) \\ &\qquad + 4\left(\frac{4n^2}{(12m^2+n^2)^3} - \frac{1}{(12m^2+n^2)^2}\right)\Bigg) \\ &= \frac{10\sqrt{3}}{\pi^3}\sum_{m,n\in\mathbb{Z}}{}' \Bigg(\frac{2(3n^2-m^2)}{(m^2+3n^2)^3} + \frac{8(m^2-12n^2)}{(m^2+12n^2)^3} \\ &\qquad + \frac{4}{(m^2+12n^2)^2} - \frac{1}{(m^2+3n^2)^2}\Bigg). \end{aligned} \tag{3.3}$$

It was proved in [3, Cor. 4.4] that the following identity holds:

$$\frac{9}{8}\sum_{m,n\in\mathbb{Z}}{}' \frac{m^2-3n^2}{(m^2+3n^2)^3} = \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{m^2-12n^2}{(m^2+12n^2)^3} + \frac{4n^2-3m^2}{(3m^2+4n^2)^3}\right). \tag{3.4}$$

Equivalently, one has that

$$\begin{aligned} \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{2(3n^2-m^2)}{(m^2+3n^2)^3} + \frac{8(m^2-12n^2)}{(m^2+12n^2)^3}\right) &= \frac{5}{2}\sum_{m,n\in\mathbb{Z}}{}' \frac{m^2-3n^2}{(m^2+3n^2)^3} \\ &\qquad + 4\sum_{m,n\in\mathbb{Z}}{}' \left(\frac{m^2-12n^2}{(m^2+12n^2)^3} + \frac{3m^2-4n^2}{(3m^2+4n^2)^3}\right) \\ &= 5L(g_{12},3) + 8L(g_{48},3), \end{aligned} \tag{3.5}$$

where the last equality is a direct consequence of Lemma 2.7 and Lemma 2.12 in [18].

Recall from Glasser and Zucker's results on lattice sums [7, Table VI] that

$$\begin{aligned} \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+3n^2)^2} &= \frac{9}{4}\zeta(2)L(\chi_{-3},2) = \frac{3\pi^2}{8}L(\chi_{-3},2), \\ \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+12n^2)^2} &= \frac{69}{64}\zeta(2)L(\chi_{-3},2) + L(\chi_{12},2)L(\chi_{-4},2) \\ &= \frac{23\pi^2}{128}L(\chi_{-3},2) + \frac{\pi^2}{6\sqrt{3}}L(\chi_{-4},2). \end{aligned} \tag{3.6}$$

Then we substitute (3.5) and (3.6) in (3.3) to get

$$n_4(26856+15300\sqrt{3}) = \frac{50\sqrt{3}}{\pi^3}L(g_{12},3)+\frac{80\sqrt{3}}{\pi^3}L(g_{48},3)+\frac{55\sqrt{3}}{16\pi}L(\chi_{-3},2)+\frac{20}{3\pi}L(\chi_{-4},2).$$

To deduce the first formula in the proposition we only need one more step, which can be done using the following well-known functional equations for L-functions of weight 3 newforms and Dirichlet L-functions:

$$\left(\frac{\sqrt{N}}{2\pi}\right)^{s}\Gamma(s)L(g_N,s) = \left(\frac{\sqrt{N}}{2\pi}\right)^{3-s}\Gamma(3-s)L(g_N,3-s),$$
$$\left(\frac{\pi}{k}\right)^{-\frac{2-s}{2}}\Gamma\left(\frac{2-s}{2}\right)L(\chi_{-k},1-s) = \left(\frac{\pi}{k}\right)^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)L(\chi_{-k},s).$$

The second formula can be shown in a similar manner by choosing $\tau_0=\sqrt{-3}/2$. Although Weber did not list an explicit value of $\mathfrak{f}_1(\sqrt{-3})$ in his book, one can find it easily using the identity $\mathfrak{f}_1(2\tau)=\mathfrak{f}(\tau)\mathfrak{f}_1(\tau)$ and the fact that $\mathfrak{f}(\sqrt{-3})=2^{\frac{1}{3}}$. Therefore, we have $s_4(\tau_0)=26856-15300\sqrt{3}$, and

$$\begin{aligned} n_4(s_4(\tau_0)) &= \frac{20\sqrt{3}}{\pi^3}\sum_{m,n\in\mathbb{Z}}{}' \left(\frac{8(3m^2-4n^2)}{(3m^2+4n^2)^3}+\frac{2(m^2-3n^2)}{(m^2+3n^2)^3}+\frac{1}{(m^2+3n^2)^2}-\frac{4}{(3m^2+4n^2)^2}\right)\\ &= \frac{20\sqrt{3}}{\pi^3}\left(-5L(g_{12},3)+8L(g_{48},3)-\frac{11\pi^2}{32}L(\chi_{-3},2)+\frac{2\pi^2}{3\sqrt{3}}L(\chi_{-4},2)\right),\end{aligned}$$

where we again use (3.4), (3.6), and the identity

$$2L(\chi_{12},2)L(\chi_{-4},2) = \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{1}{(m^2+12n^2)^2}-\frac{1}{(3m^2+4n^2)^2}\right)$$

(see [18, Lem. 2.6]). □

Besides the formulas proved in Proposition 3.1, at the end of this paper we will tabulate all three-variable Mahler measure formulas that we found from numerical computations. Though these formulas can be verified by computing the involved values to a high degree of accuracy, most of them are still conjectural; i.e., they have not been proved rigorously. The proved formulas, except the two formulas above, were established in [14] and [18]. In Table 1-3, we use the following shorthand notations:

$$M_N := L'(g_N,0), \qquad d_k := L'(\chi_{-k},-1),$$

where g_N is a normalized newform with rational Fourier coefficients in $S_3(\Gamma_0(N),\chi_{-N})$. If there are more than one such newforms, we shall distinguish them using superscripts.

4. Functional equations in the three-variable case

One has seen from [10] that $m_2(t)$ satisfies some functional equations, which can be applied in establishing new Mahler measure formulas as shown in Section 2. This section aims to derive a functional equation for three-variable Mahler measures. We will show that

Theorem 4.1. *If $t \in \mathbb{C}\backslash\{0\}$ and $|t|$ is sufficiently small, then*

$$n_2\left(\frac{16}{t(1-t)}\right) = 9n_2\left(\frac{4(1+\sqrt{1-t})^6}{t^2\sqrt{1-t}}\right) + 4n_2\left(\frac{-2^{10}(1+\sqrt{1-t})^6\sqrt{1-t}}{t^4}\right)$$
$$- n_2\left(\frac{-16(1-t)^2}{t}\right) - 8n_2\left(\frac{2(1+\sqrt[4]{1-t})^{12}}{t(1-\sqrt{1-t})^3\sqrt[4]{1-t}}\right).$$

Proof. The proof requires some preliminary results from [14, Thm. 2.3] and Ramanujan's theory of elliptic functions. Following notations in [14], we let

$$G(q) := \mathrm{Re}\left(-\log(q) + 240\sum_{n=1}^{\infty} n^2 \log(1-q^n)\right),$$
$$\chi(q) := \prod_{n=0}^{\infty}\left(1+q^{2n+1}\right).$$

Recall from Rogers' result that if $|q|$ is sufficiently small, then the following matrix equation holds:

$$\begin{pmatrix} G(q) \\ G(-q) \\ G(q^2) \end{pmatrix} = \begin{pmatrix} -19 & -4 & 12 \\ -4 & -19 & 12 \\ -3 & -3 & 4 \end{pmatrix} \begin{pmatrix} n_2(s_2(q)) \\ n_2(s_2(-q)) \\ 2n_2\left(s_2\left(q^2\right)\right) - n_2\left(s_2\left(-q^2\right)\right) \end{pmatrix}.$$

Expressing $G(q^2)$ in two different ways, one finds that

$$n_2(s_2(q)) = 9n_2\left(s_2\left(q^2\right)\right) + 4n_2\left(s_2\left(-q^4\right)\right) - n_2(s_2(-q)) - 8n_2\left(s_2\left(q^4\right)\right). \tag{4.2}$$

Now let

$$z_2(t) = {}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix}; t\right), \qquad y_2(t) = \frac{\pi\, {}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix}; 1-t\right)}{{}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix}; t\right)}, \qquad q_2(t) = e^{-y_2}.$$

Note that $q_2(t)$ defined above is sometimes called the *signature* 2 *elliptic nome*. It is known from [1, §17] that the following identities hold:

$$\chi(q_2) = 2^{1/6}\left(\frac{q_2}{t(1-t)}\right)^{1/24},$$
$$\chi(-q_2) = 2^{1/6}(1-t)^{1/12}\left(\frac{q_2}{t}\right)^{1/24},$$
$$\chi(-q_2^2) = 2^{1/3}(1-t)^{1/24}\left(\frac{q_2}{t}\right)^{1/12}.$$

Moreover, we can deduce formulas for $\chi(q_2^2), \chi(q_2^4)$, and $\chi(-q_2^4)$ from the identities above using a process called *obtaining a formula by duplication*; that is, if we have $\Omega(t, q_2, z_2) = 0$, then

$$\Omega\left(\left(\frac{1-\sqrt{1-t}}{1+\sqrt{1+t}}\right)^2, q_2^2, \frac{z_2(1+\sqrt{1-t})}{2}\right) = 0.$$

Therefore, by some manipulation, we find that

$$\chi^{24}(q_2^2) = \frac{4(1+\sqrt{1-t})^6}{t^2\sqrt{1-t}} q_2^2,$$
$$\chi^{24}(q_2^4) = \frac{2(1+\sqrt[4]{1-t})^{12}}{t(1-\sqrt{1-t})^3\sqrt[4]{1-t}} q_2^4,$$
$$\chi^{24}(-q_2^4) = \frac{2^{10}(1+\sqrt{1-t})^6\sqrt{1-t}}{t^4} q_2^4.$$

The theorem then follows immediately from these identities and (4.2). □

As an application of Theorem 4.1, we can deduce a five-term relation

$$\begin{aligned} n_2(64) = 9n_2\left(280+198\sqrt{2}\right) + 4n_2\left(-143360-101376\sqrt{2}\right) \\ - n_2(-8) - 8n_2\left(71704+50688\sqrt{2}+60282\sqrt[4]{2}+42633\sqrt[4]{8}\right) \end{aligned}$$

by letting $t = 1/2$. It would be interesting to see if each term in the equation above is related to special L-values. It turns out that only a partial answer can be given here. First, it was rigorously proved in [18, Thm. 1.2] that $n_2(64) = 8L'(g_{16}, 0)$, where $g_{16}(\tau) = \eta^6(4\tau) \in S_3(\Gamma_0(16), \chi_{-4})$. Then, using the hypergeometric representation of $n_2(s)$ given in [14, Prop. 2.2] and integral representations of L-functions, we are able to verify numerically that the following formulas hold:

$$n_2(-8) \stackrel{?}{=} 4L'(g_{16},0) + L'(\chi_{-4},-1),$$
$$n_2\left(280+198\sqrt{2}\right) \stackrel{?}{=} \frac{1}{8}(36L'(g_{16},0) + 4L'(g_{64},0) + 13L'(\chi_{-4},-1) + 4L'(\chi_{-8},-1)),$$

where $g_{64}(\tau)$ is the normalized newform of weight 3 and level 64 with rational Fourier coefficients. Nevertheless, no similar evidence for the remaining two terms has been found. From the previous examples and numerical observations exhibited at the end of this paper, it is not unreasonable to conjecture that

$$n_2\left(-143360-101376\sqrt{2}\right) \text{ and } n_2\left(71704+50688\sqrt{2}+60282\sqrt[4]{2}+42633\sqrt[4]{8}\right)$$

involve two and four modular L-values, respectively, corresponding to weight 3 newforms of higher level. However, we are still unable to find the L-values of the newforms that are likely to be our possible candidates.

It is also possible to obtain a functional equation for $n_4(s)$ defined in Section 3 using similar arguments above. Again, we see from [14] that for $|q|$ sufficiently small

$$\begin{pmatrix} G(q) \\ G(-q) \\ G(q^2) \end{pmatrix} = \begin{pmatrix} -5 & -2 & 4 \\ -2 & -5 & 4 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} n_4(s_4(q)) \\ n_4(s_4(-q)) \\ n_4(s_4(q^2)) \end{pmatrix}.$$

Hence we find that

$$n_4(s_4(q)) = 7n_4\left(s_4\left(q^2\right)\right) + 2n_4\left(s_4\left(-q^2\right)\right) - n_4(s_4(-q)) - 4n_4\left(s_4\left(q^4\right)\right).$$

To express $s_4(q), s_4(-q), s_4(q^2), s_4(-q^2)$, and $s_4(q^4)$ in terms of algebraic functions of some parameter we need the Ramanujan's theory of signature 4. (See [2] for references.) However, the results we found are quite complicated because of multiple radical terms, so we do not include them here.

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τ	$s_2(\tau)$	$n_2(s_2(\tau))$	Proved?
$\frac{\sqrt{-1}}{2}$	64	$8M_{16}$	Yes
$\frac{1+\sqrt{-1}}{2}$	-8	$4M_{16}+d_4$	No
$\frac{\sqrt{-4}}{2}$	$280+198\sqrt{2}$	$\frac{1}{8}\left(36M_{16}+4M_{64}+13d_4+4d_8\right)$	No
$\frac{2+\sqrt{-1}}{4}$	$280-198\sqrt{2}$	$\frac{1}{2}\left(36M_{16}-4M_{64}-13d_4+4d_8\right)$	No
$\frac{1+\sqrt{-4}}{2}$	-512	$M_{64}+d_8$	No
$\frac{\sqrt{-2}}{2}$	$56+40\sqrt{2}$	$\frac{1}{4}\left(60M_8+4M_{32}+4d_4+d_8\right)$	No
$\frac{2+\sqrt{-2}}{4}$	$56-40\sqrt{2}$	$\frac{1}{2}\left(60M_8-4M_{32}+4d_4-d_8\right)$	No
$\frac{1+\sqrt{-2}}{2}$	-64	$2\left(M_{32}+d_4\right)$	No
$\frac{\sqrt{-3}}{2}$	256	$\frac{4}{3}\left(M_{48}+2d_4\right)$	Yes
$\frac{1+\sqrt{-3}}{4}$	16	$8M_{12}$	No
$\frac{3+\sqrt{-3}}{6}$	$-104+60\sqrt{3}$	$\frac{1}{2}\left(4M_{48}-36M_{12}+15d_3-8d_4\right)$	No
$\frac{1+\sqrt{-3}}{2}$	$-104-60\sqrt{3}$	$\frac{1}{6}\left(4M_{48}+36M_{12}+15d_3+8d_4\right)$	No
$\frac{\sqrt{-6}}{2}$	$568+384\sqrt{2}$ $+336\sqrt{3}+216\sqrt{6}$	$\frac{1}{24}\Big(60M_{24}^{(1)}+12M_{24}^{(2)}+4M_{96}^{(1)}+4M_{96}^{(2)}$ $+60d_3+24d_4+8d_8+d_{24}\Big)$	No
$\frac{6+\sqrt{-6}}{12}$	$568+384\sqrt{2}$ $-336\sqrt{3}-216\sqrt{6}$	$\frac{1}{4}\Big(60M_{24}^{(1)}+12M_{24}^{(2)}-4M_{96}^{(1)}-4M_{96}^{(2)}$ $-60d_3+24d_4+8d_8-d_{24}\Big)$	No
$\frac{\sqrt{-6}}{6}$	$568-384\sqrt{2}$ $+336\sqrt{3}-216\sqrt{6}$	$\frac{1}{12}\Big(60M_{24}^{(1)}-12M_{24}^{(2)}+4M_{96}^{(1)}-4M_{96}^{(2)}$ $+60d_3+24d_4-8d_8-d_{24}\Big)$	No
$\frac{-2+\sqrt{-6}}{10}$	$568-384\sqrt{2}$ $-336\sqrt{3}+216\sqrt{6}$	$\frac{1}{12}\Big(60M_{24}^{(1)}-12M_{24}^{(2)}-4M_{96}^{(1)}+4M_{96}^{(2)}$ $+60d_3-24d_4+8d_8-d_{24}\Big)$	No
$\frac{3+\sqrt{-6}}{6}$	$-1088+768\sqrt{2}$	$M_{96}^{(1)}-M_{96}^{(2)}-6d_4+2d_8$	No
$\frac{1+\sqrt{-6}}{2}$	$-1088-768\sqrt{2}$	$\frac{1}{3}\left(M_{96}^{(1)}+M_{96}^{(2)}+6d_4+2d_8\right)$	No
$\frac{\sqrt{-7}}{2}$	4096	$\frac{4}{7}\left(M_{112}+8d_4\right)$	No
$\frac{3+\sqrt{-7}}{8}$	1	$8M_7$	No
$\frac{7+\sqrt{-7}}{14}$	$-2024+765\sqrt{7}$	$\frac{1}{2}\left(4M_{112}-384M_7-32d_4+11d_7\right)$	No
$\frac{1+\sqrt{-7}}{2}$	$-2024-765\sqrt{7}$	$\frac{1}{14}\left(4M_{112}+384M_7+32d_4+11d_7\right)$	No

TABLE 1. Special values of $n_2(s)$

τ	$s_3(\tau)$	$n_3(s_3(\tau))$	Proved?
$\frac{1+\sqrt{-2}}{3}$	8	$15M_8$	No
$\frac{\sqrt{-3}}{3}$	108	$15M_{12}$	Yes
$\frac{\sqrt{-6}}{3}$	216	$\frac{15}{4}\left(M_{24}^{(2)}+d_3\right)$	Yes
$\frac{\sqrt{-9}}{3}$	$288+168\sqrt{3}$	$\frac{5}{12}\left(3M_{36}^{(2)}+3M_{36}^{(1)}+6d_3+4d_4\right)$	No
$\frac{1+\sqrt{-1}}{2}$	$288-168\sqrt{3}$	$\frac{5}{6}\left(3M_{36}^{(2)}-3M_{36}^{(1)}-6d_3+4d_4\right)$	No
$\frac{\sqrt{-12}}{3}$	1458	$\frac{15}{8}\left(9M_{12}+2d_4\right)$	Yes
$\frac{\sqrt{-15}}{3}$	3375	$\frac{3}{5}\left(20M_{15}^{(2)}+13d_3\right)$	No
$\frac{\sqrt{-18}}{3}$	$3704+1456\sqrt{6}$	$\frac{5}{24}\left(3M_{72}+72M_8+18d_3+4d_8\right)$	No
$\frac{\sqrt{-2}}{2}$	$3704-1456\sqrt{6}$	$\frac{5}{12}\left(3M_{72}-72M_8-18d_3+4d_8\right)$	No
$\frac{\sqrt{-21}}{3}$	$7344+2808\sqrt{7}$	$\frac{15}{28}\left(M_{84}^{(2)}+M_{84}^{(4)}+4d_4+2d_7\right)$	No
$\frac{3+\sqrt{-21}}{6}$	$7344-2808\sqrt{7}$	$\frac{15}{14}\left(M_{84}^{(2)}-M_{84}^{(4)}-4d_4+2d_7\right)$	No
$\frac{\sqrt{-24}}{3}$	$14310+8262\sqrt{3}$	$\frac{15}{32}\left(7M_{24}^{(2)}+M_{96}^{(2)}+11d_3+6d_4\right)$	No
$\frac{-3+\sqrt{-6}}{2}$	$14310-8262\sqrt{3}$	$\frac{15}{8}\left(7M_{24}^{(2)}-M_{96}^{(2)}+11d_3-6d_4\right)$	No
$\frac{\sqrt{-30}}{3}$	$48168+15120\sqrt{10}$	$\frac{3}{40}\left(5M_{120}^{(2)}+5M_{120}^{(4)}+5d_{15}+2d_{24}\right)$	No
$\frac{6+\sqrt{-30}}{6}$	$48168-15120\sqrt{10}$	$\frac{3}{20}\left(5M_{120}^{(2)}-5M_{120}^{(4)}+5d_{15}-2d_{24}\right)$	No

TABLE 2. Special values of $n_3(s)$

τ	$s_4(\tau)$	$n_4(s_4(\tau))$	Proved?
$\frac{\sqrt{-2}}{2}$	256	$40M_8$	Yes
$\frac{\sqrt{-8}}{2}$	$3656+2600\sqrt{2}$	$\frac{5}{8}\left(4M_{32}+28M_8+4d_4+d_8\right)$	Yes
$\frac{1+\sqrt{-2}}{2}$	$3656-2600\sqrt{2}$	$\frac{5}{4}\left(4M_{32}-28M_8+4d_4-d_8\right)$	Yes
$\frac{\sqrt{-12}}{2}$	$26856+15300\sqrt{3}$	$\frac{5}{12}\left(4M_{48}+20M_{12}+11d_3+8d_4\right)$	No
$\frac{\sqrt{-3}}{2}$	$26856-15300\sqrt{3}$	$\frac{5}{6}\left(4M_{48}-20M_{12}-11d_3+8d_4\right)$	No
$\frac{1+\sqrt{-3}}{2}$	-144	$\frac{10}{3}\left(4M_{12}+d_3\right)$	No
$\frac{\sqrt{-4}}{2}$	648	$\frac{5}{2}\left(4M_{16}+d_4\right)$	Yes
$\frac{\sqrt{-16}}{2}$	$143208+101574\sqrt{2}$	$\frac{5}{16}\left(4M_{64}+20M_{16}+9d_4+4d_8\right)$	No
$\frac{1+\sqrt{-4}}{2}$	$143208-101574\sqrt{2}$	$\frac{5}{8}\left(4M_{64}-20M_{16}-9d_4+4d_8\right)$	No
$\frac{1+\sqrt{-5}}{2}$	-1024	$\frac{8}{5}\left(5M_{20}^{(1)}+2d_4\right)$	No
$\frac{\sqrt{-6}}{2}$	2304	$\frac{20}{3}\left(M_{24}^{(1)}+d_3\right)$	Yes
$\frac{\sqrt{-24}}{2}$	$1207368+853632\sqrt{2}$ $+697680\sqrt{3}+493272\sqrt{6}$	$\frac{5}{48}\big(4M_{96}^{(1)}+4M_{96}^{(2)}+28M_{24}^{(1)}+12M_{24}^{(2)}$ $+28d_3+24d_4+8d_8+d_{24}\big)$	No
$\frac{1+\sqrt{-6}}{2}$	$1207368+853632\sqrt{2}$ $-697680\sqrt{3}-493272\sqrt{6}$	$\frac{5}{24}\big(4M_{96}^{(1)}+4M_{96}^{(2)}-28M_{24}^{(1)}-12M_{24}^{(2)}$ $-28d_3+24d_4+8d_8-d_{24}\big)$	No
$\frac{\sqrt{-6}}{4}$	$1207368-853632\sqrt{2}$ $+697680\sqrt{3}-493272\sqrt{6}$	$\frac{5}{16}\big(4M_{96}^{(1)}-4M_{96}^{(2)}+28M_{24}^{(1)}-12M_{24}^{(2)}$ $-28d_3-24d_4+8d_8+d_{24}\big)$	No
$\frac{2+\sqrt{-6}}{4}$	$1207368-853632\sqrt{2}$ $-697680\sqrt{3}+493272\sqrt{6}$	$\frac{5}{12}\big(-4M_{96}^{(1)}+4M_{96}^{(2)}+28M_{24}^{(1)}-12M_{24}^{(2)}$ $+28d_3-24d_4+8d_8-d_{24}\big)$	No
$\frac{\sqrt{-28}}{2}$	$8292456+3132675\sqrt{7}$	$\frac{5}{28}\left(4M_{112}+224M_7+32d_4+7d_7\right)$	No
$\frac{\sqrt{-7}}{2}$	$8292456-3132675\sqrt{7}$	$\frac{5}{14}\left(4M_{112}-224M_7+32d_4-7d_7\right)$	No
$\frac{1+\sqrt{-7}}{2}$	-3969	$\frac{10}{7}\left(40M_7+d_7\right)$	No
$\frac{1+\sqrt{-9}}{2}$	-12288	$\frac{40}{9}\left(M_{36}^{(1)}+2d_3\right)$	No
$\frac{\sqrt{-10}}{2}$	20736	$\frac{4}{5}\left(5M_{40}^{(1)}+2d_8\right)$	Yes
$\frac{1+\sqrt{-13}}{2}$	-82944	$\frac{40}{13}\left(M_{52}^{(1)}+2d_4\right)$	No
$\frac{3+\sqrt{-15}}{6}$	$\frac{-192303+85995\sqrt{5}}{2}$	$\frac{1}{5}\left(160M_{15}^{(1)}-120M_{15}^{(2)}-88d_3+5d_{15}\right)$	No
$\frac{1+\sqrt{-15}}{2}$	$\frac{-192303-85995\sqrt{5}}{2}$	$\frac{1}{15}\left(160M_{15}^{(1)}+120M_{15}^{(2)}+88d_3+5d_{15}\right)$	No
$\frac{\sqrt{-18}}{2}$	614656	$\frac{40}{3}(5M_8+d_3)$	Yes
$\frac{3+\sqrt{-21}}{6}$	$-893952+516096\sqrt{3}$	$\frac{20}{7}\left(M_{84}^{(3)}-M_{84}^{(4)}+8d_3-4d_4\right)$	No
$\frac{1+\sqrt{-21}}{2}$	$-893952-516096\sqrt{3}$	$\frac{20}{21}\left(M_{84}^{(3)}+M_{84}^{(4)}+8d_3+4d_4\right)$	No

TABLE 3. Special values of $n_4(s)$

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